Some Analysis Methods for Rotating Systems with Periodic Coefficients

John Dugundji* and John H. Wendell†
Massachusetts Institute of Technology, Cambridge, Massachusetts

This paper reviews two of the more common procedures for analyzing the stability and forced response of equations with periodic coefficients, namely, the use of Floquet methods and the use of multiblade coordinate and harmonic balance methods. The use of rotating coordinates and perturbation methods is also briefly discussed. The analysis procedures of these periodic coefficient systems are compared with those of the more familiar constant coefficient systems.

Introduction

IN dynamic analyses of rotating wind-turbine or helicopter systems, equations of motion with periodic coefficients are frequently encountered. Unlike systems with constant coefficients whose analysis techniques are well known and familiar, the analysis of these periodic coefficient equations are somewhat less familiar. This paper reviews two of the more common procedures for analyzing the stability and response of these periodic coefficient equations: the use of Floquet methods and the use of multiblade coordinate and harmonic balance methods. Also, the use of rotating coordinates is discussed. A third procedure involving the use of perturbation methods will be mentioned, but not discussed in detail.

To put things in proper perspective and to make comparisons, the constant coefficient systems are briefly reviewed first. This paper is an extension of Appendices A-D of a report by the authors.[1] A good treatment of similar material can be found in Ref. 2.

Constant Coefficient Systems

Given a system of N linear differential equations with constant coefficients,

$$\underline{M}\ddot{\underline{q}}+\underline{B}\dot{\underline{q}}+\underline{K}\underline{q}=\underline{F}(t) \tag{1}$$

where $\underline{M}$, $\underline{B}$, and $\underline{K}$ are square matrices of order $N\times N$, while $\underline{q}$ and $\underline{F}(t)$ are column matrices of order $N\times 1$. These can be rearranged as

$$\begin{bmatrix} \underline{M} & 0 \\ 0 & \underline{M} \end{bmatrix}\begin{Bmatrix} \dot{\underline{q}} \\ \ddot{\underline{q}} \end{Bmatrix}=\begin{bmatrix} 0 & \underline{M} \\ -\underline{K} & -\underline{B} \end{bmatrix}\begin{Bmatrix} \underline{q} \\ \dot{\underline{q}} \end{Bmatrix}=\begin{Bmatrix} 0 \\ \underline{F} \end{Bmatrix} \tag{2}$$

Then, multiplying through by the inverse of the mass matrix gives $2N$ first-order equations,

$$\dot{\underline{y}}-\underline{A}\underline{y}=\underline{G} \tag{3}$$

where $\underline{A}$ is a square matrix of order $2N\times 2N$, while $\underline{y}$ and $\underline{G}$ are column matrices of order $2N\times 1$ given by

$$\underline{A}=\begin{bmatrix} 0 & \underline{1} \\ -\underline{M}^{-1}\underline{K} & -\underline{M}^{-1}\underline{B} \end{bmatrix},\quad \underline{y}=\begin{Bmatrix} \underline{q} \\ \dot{\underline{q}} \end{Bmatrix},\quad \underline{G}=\begin{Bmatrix} 0 \\ \underline{M}^{-1}\underline{F} \end{Bmatrix} \tag{4}$$

Equation (3) is valid providing the mass $\underline{M}$ is not singular, which is usually the case with physical systems.

Stability

To investigate stability, set $\underline{F}=0$ (which gives $\underline{G}=0$) in Eq. (3) to obtain a set of homogeneous equations and then seek the exponential solutions of the form $\underline{y}=\underline{y}e^{pt}$. Placing these into Eq. (3) leads to the standard eigenvalue problem,

$$\underline{A}\underline{y}=p\underline{y} \tag{5}$$

Eigenvalues p_k of the matrix $\underline{A}$ can be obtained by standard numerical eigenvalue routines. If any eigenvalue p_k is positive real or has a positive real part, the system represented by Eq. (3) or equivalently by Eq. (1) is unstable. It should be remarked that for certain simple specialized cases, other methods can also be used to investigate stability directly from Eq. (1), such as expanding the characteristic determinant, applying Routh-Hurwitz criteria or applying the Kelvin-Tait-Chetaev theorem.[3]

Forced Response

Under steady-state conditions, the forces $\underline{F}(t)$ on a rotating system tend to occur periodically in multiples of the rotation frequency Ω. The force for a particular frequency $\omega_m=m\Omega$ can then be expressed in the form,

$$\underline{F}(t)=Re(\underline{F}e^{i\omega_m t})=\underline{F}_R\cos\omega_m t-\underline{F}_I\sin\omega_m t \tag{6}$$

The response $\underline{q}(t)$ is similarly of the form,

$$\underline{q}(t)=Re(\underline{q}e^{i\omega_m t})=\underline{q}_R\cos\omega_m t-\underline{q}_I\sin\omega_m t \tag{7}$$

Placing Eqs. (6) and (7) into the basic Eq. (1) and matching sine and cosine terms gives a set of $2N\times 2N$ real equations,

$$\begin{bmatrix} \underline{G} & \underline{H} \\ -\underline{H} & \underline{G} \end{bmatrix}\begin{Bmatrix} \underline{q}_R \\ \underline{q}_I \end{Bmatrix}=\begin{Bmatrix} \underline{F}_R \\ \underline{F}_I \end{Bmatrix} \tag{8}$$

which has the matrix elements,

$$\underline{G}=\underline{K}-\omega_m^2\underline{M},\qquad \underline{H}=\omega_m\underline{B} \tag{9}$$

Given the amount of the mth harmonic force present $\underline{F}_R^{(m)}$ and $\underline{F}_I^{(m)}$, Eq. (8) can be solved by simple inversion to find the

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*Professor, Department of Aeronautics and Astronautics. Member AIAA.
†Graduate Student, Department of Aeronautics and Astronautics. Member AIAA.

response $\underline{q}_R^{(m)}$ and $\underline{q}_I^{(m)}$ for each harmonic. Then, all of the harmonics can be summed to give the total periodic response as,

$$\underline{q}(t) = \sum_{m=0}^{N} \underline{q}_R^{(m)} \cos\omega_m t - \sum_{m=0}^{N} \underline{q}_I^{(m)} \sin\omega_m t \tag{10}$$

Finding the response $\underline{q}(t)$ this way rather than by direct numerical integration allows assessment of the effects of a particular harmonic on the resulting response of the system. Of course, if numerical integration is used, a Fourier analysis can be made of the resulting response to obtain the effects of different harmonics.

Floquet Methods

Assume the coefficients $\underline{M}$, $\underline{B}$, and $\underline{K}$ in Eq. (1) or equivalently the coefficients $\underline{A}$ in Eq. (3) vary periodically in time, rather than being constants. For illustrating Floquet methods, it will be convenient to use the first-order representation, namely $2N$ equations of the form,

$$\dot{\underline{y}} - \underline{A}(t)\underline{y} = \underline{G}(t) \tag{11}$$

where $\underline{A}(t)$ and $\underline{G}(t)$ are periodic over an interval T.

Stability

The Floquet stability analysis described here follows that given by Peters and Hohenemser.[4] Some other good discussions of Floquet methods are given by Johnson,[2] Nayfeh and Mook,[5] and Brockett.[6] To investigate stability, one sets $\underline{G}=0$ in Eq. (11) to obtain homogeneous equations. The Floquet theorem states the solution of Eq. (11) with $\underline{G}=0$ is of the form

$$\underline{y}(t) = \underline{B}(t)\{C_k e^{p_k t}\} \tag{12}$$

where $\underline{y}(t)$ and $\{C_k e^{p_k t}\}$ are $2N\times 1$ column matrices, and $\underline{B}(t)$ is a $2N\times 2N$ square matrix periodic over period T, that is, $\underline{B}(T)=\underline{B}(0)$. From the above, one can express

$$\underline{y}(0) = \underline{B}(0)\{C_k\} \tag{13}$$

$$\underline{y}(T) = \underline{B}(T)\{C_k e^{p_k T}\} = \underline{B}(0)\{C_k e^{p_k T}\} \tag{14}$$

Also, one can express $\underline{y}(T)$ as,

$$\underline{y}(T) = \underbrace{[\underline{y}^{(1)}\underline{y}^{(2)}\ldots]}_{[Q]} \begin{Bmatrix} y_1(0) \\ y_2(0) \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{15}$$

where $\underline{y}^{(1)}$ is the solution at $t=T$ of Eq. (11) with $\underline{G}=0$ for the initial conditions $y_1=1$ and all remaining $y_i(0)=0$, $\underline{y}^{(2)}$ is the solution for $y_2(0)=1$ and all remaining $y_i(0)=0$, etc. The square matrix $[Q]$ is called the "transition matrix." Equating Eq. (15) to Eq. (14) and introducing Eq. (13) gives

$$[Q][\{B(0)\}_1 C_1 + \{B(0)\}_2 C_2 + \ldots] = \{B(0)_1\} C_1 e^{p_1 T} + \ldots \tag{16}$$

Since C_k are independent, one must have

$$[Q]\{B(0)\}_k = \lambda_k \{B(0)\}_k \tag{17}$$

where $\lambda_k = e^{p_k T}$ are the eigenvalues of the $[Q]$ matrix. One then has the relation

$$p_k = (1/T)\ln\lambda_k = \alpha_k + i\omega_k \tag{18}$$

from which the real and imaginary parts of the stability exponent p_k are given as

$$\alpha_k = (1/T)\ln|\lambda_k| = (1/2T)\ln[(\lambda_k)_R^2 + (\lambda_k)_I^2] \tag{19a}$$

$$\omega_k = (1/T)\tan^{-1}[(\lambda_k)_I/(\lambda_k)_R] \tag{19b}$$

The real part α_k is a measure of the growth or decay of the response, as can be seen from Eq. (12). Values of $\alpha_k>0$ (or equivalently $|\lambda_k|>1$) indicate instability. The imaginary part ω_k represents the frequency. However, because $\tan^{-1}$ is multivalued, ω_k can only be obtained as a basic frequency plus or minus any integer multiple of $2\pi/T$. This implies that part of the periodicity of the motion is in the root ω_k and part in the periodic function $B(t)$, which adjusts itself accordingly. Various choices for ω_k can be made, ranging from merely choosing the principal value (no multiple of $2\pi/T$) to choosing a frequency expected from physical considerations, see Refs. 2 and 4. However, the actual motion and actual frequency content corresponding to the kth root p_k can always be obtained numerically by setting $C_k=1$ and all other remaining $C_i=0$ in Eqs. (12) and (13). Then, using the kth eigenvector $\{B(0)\}_k$ from Eq. (17) as an initial condition, Eq. (11) is solved with $G=0$ by numerical integration techniques for the resultant motion. This resultant motion $\underline{y}_k(t)$ for the kth root can also be expressed as,

$$\underline{y}_k(t) = [Q(t)]\{B(0)\}_k \tag{20}$$

where $[Q(t)]$ represents the transition matrix at every time interval t instead of only at the end point T. To find the periodic function $\{B(t)\}_k$ corresponding to the kth root p_k, merely multiply the $\underline{y}_k$ obtained in Eq. (20) above by $e^{-p_k t}$.

Summarizing: To check for stability of a system of linear equations with periodic coefficients, obtain the eigenvalues λ_k of the "transition matrix" $[Q]$. If $|\lambda_k|>1$, one has instability. The traditional stability exponent p_k is related to λ_k through Eqs. (18-20). Three remarks on this procedure should be noted:

1) The "transition matrix" $[Q]$ can be formed by solving either the first-order equations, Eq. (11) with $\underline{G}=0$, or the second-order equations, Eq. (1) with $\underline{F}=0$ and periodic coefficients, whichever is more convenient for the integration scheme.

2) To obtain the "transition matrix" $[Q]$, employ either a "$2N$-pass approach" (integrating the equations of motion $2N$ times to cover all initial conditions) or a "single-pass approach" (integrating the equations once, but keeping the results for all initial conditions simultaneously). Both Friedmann et al.[7] and a later study by Gaonkar et al.[8] favor the numerical efficiency of the "single-pass approach."

3) The preceding Floquet procedure will still apply even if the equations have constant coefficients. However, for such cases it is usually easier to form the matrix $\underline{A}$ given by Eq. (4) and obtain its eigenvalues p_k rather than to form the "transition matrix" $[Q]$ and obtain its eigenvalues λ_k.

Forced Response

Solutions of Eq. (11), or equivalently Eq. (1) with periodic coefficients, can be obtained by direct numerical integration using some convenient integration scheme. By proper choice of the initial conditions, all transients can be eliminated from the response and the desired steady-state dynamic response thus obtained by integrating through only one period T, instead of the very large number usually required to reach steady-state for lightly damped systems. A procedure for finding the proper initial conditions is given below.

Solutions of Eq. (11) are of the general form,

$$\underline{y}(t)=\underline{y}_H(t)+\underline{y}_p(t) \tag{21}$$

where $\underline{y}_H(t)$ is the homogeneous solution and $\underline{y}_p(t)$ is the particular solution. A complete solution of Eq. (11) can be obtained numerically for any given set of initial conditions. Call this solution $\underline{y}_E(t)$. One can add any number of additional homogeneous solutions $\Delta\underline{y}_H(t)$ having different initial conditions to this solution $\underline{y}_E(t)$. This would give new additional solutions $\underline{y}(t)$ to Eq. (11),

$$\underline{y}(t)=\underline{y}_E(t)+\Delta\underline{y}_H(t) \tag{22}$$

which would now have different initial conditions than those of the original $\underline{y}_E(t)$.

All of the homogeneous solutions of Eq. (11) can be found by solving Eq. (11) with $\underline{G}=0$ a total of $2N$ times, subject to the initial conditions $y_1=1$ and all remaining $y_i=0$, then $y_2=1$ and all remaining $y_i=0$, etc. In fact, this was done earlier to investigate stability and resulted in the $2N$ homogeneous solutions $\underline{y}^{(1)}(t)$, $\underline{y}^{(2)}(t)$, etc., respectively. Thus, one may write

$$\Delta\underline{y}_H(t)=\underbrace{[\underline{y}^{(1)}(t)\,\underline{y}^{(2)}(t)\ldots]}_{\substack{\|\\ [Q(t)]}}\begin{Bmatrix} C_1\\ C_2\\ \cdot\\ \cdot\\ \cdot \end{Bmatrix} \tag{23}$$

where $[Q(t)]$ is the transition matrix at any instant of time, and C_1, C_2, ... are $2N$ arbitrary constants. The new solution [Eq. (22)] can be rewritten as

$$\underline{y}(t)=\underline{y}_E(t)+[Q(t)]\underline{C} \tag{24}$$

For a periodic solution over period $T=2\pi/\Omega$, $\underline{y}(T)$ must equal $\underline{y}(0)$. Placing Eq. (24) into this condition and solving for the arbitrary constants $\underline{C}$ gives,

$$\underline{y}_E(T)+[Q(T)]\underline{C}=\underline{y}_E(0)+[Q(0)]\underline{C}$$

$$\underline{C}=[\underline{I}-[Q]]^{-1}\{\underline{y}_E(T)-\underline{y}_E(0)\} \tag{25}$$

where it was noted that $[Q(0)]=\underline{I}$, and $[Q(T)]=[Q]$ is the "transition matrix" found earlier for the stability investigation. Placing these values of $\underline{C}$ back into Eq. (24), the initial conditions for insuring a periodic solution become

$$\underline{y}(0)=\underline{y}_E(0)+[\underline{I}-\underline{Q}]^{-1}\{\underline{y}_E(T)-\underline{y}_E(0)\} \tag{26}$$

The basic equation (11) can then be solved numerically with these initial conditions to obtain a periodic solution over one period. It should be noted that if the initial conditions chosen for $\underline{y}_E(t)$ were $\underline{y}_E(0)=0$, the result would simply be

$$\underline{y}(0)=[\underline{I}-\underline{Q}]^{-1}\underline{y}_E(T) \tag{27}$$

where $\underline{y}_E(T)$ is now the numerical response of the complete Eq. (11) at $t=T$ for zero initial conditions. This is a particularly convenient form for finding the initial conditions for periodic solutions.

An alternative form for determining the proper initial conditions for periodic solutions has been proposed by Friedmann and his co-workers[9,10] in their work on wind turbines, namely,

$$\underline{y}(0)=[\underline{I}-\underline{Q}]^{-1}\underline{Q}\int_0^T[Q(t)]^{-1}\underline{G}(t)\,\mathrm{d}t \tag{28}$$

This is similar to Eq. (27), but does not use $\underline{y}_E$. It seems easier to obtain $\underline{y}_E(T)$ with initial conditions $\underline{y}_E(0)=0$ and use Eq. (27), rather than obtaining $[Q(t)]$ at every time interval t and performing the indicated operations required by Eq. (28).‡

The general procedure described by Eqs. (21-27) may be extended to deal also with nonlinear equations,

$$\dot{\underline{y}}-\underline{A}(t)\underline{y}=\underline{G}(t,\underline{y},\dot{\underline{y}}) \tag{29}$$

where the right-hand side now contains nonlinear functions of the coordinates. An iterative variation of the previous linear procedure to obtain the initial conditions for periodic solutions of nonlinear equations is used by the MOSTAS code.[12] The procedure is as follows. First, a numerical solution $\underline{y}_E(t)$ is obtained to the nonlinear Eq. (29) for some estimate of the initial conditions $\underline{y}_E(0)$. Then each of the $2N$ elements of $\underline{y}_E(0)$ is perturbed a small amount ϵ_i and the resulting $2N$ solutions are obtained. This involves solving the nonlinear Eq. (29) subject to the initial conditions,

$$\underline{y}_E(0)+\begin{Bmatrix}\epsilon_1\\0\\0\\ \cdot\\ \cdot\\ \cdot\end{Bmatrix},\quad \underline{y}_E(0)+\begin{Bmatrix}0\\ \epsilon_2\\0\\ \cdot\\ \cdot\\ \cdot\end{Bmatrix},\quad \text{etc.} \tag{30}$$

and will result in $2N$ responses of the form

$$\underline{y}^{(i)}(t)=\underline{y}_E(t)+\Delta\underline{y}_E^{(i)}(t) \tag{31}$$

where $\Delta\underline{y}_E^{(i)}(t)$ represents the effect of each perturbation ϵ_i and is found by *subtracting* $\underline{y}_E(t)$ from each of the $2N$ resulting responses $\underline{y}^{(i)}(t)$. One can then express the total solution approximately as,

$$\underline{y}(t)=\underline{y}_E(t)+\underbrace{\left[\frac{1}{\epsilon_1}\Delta\underline{y}_E^{(1)},\frac{1}{\epsilon_2}\Delta\underline{y}_E^{(2)},\ldots\right]}_{\substack{\|\\ [Q]}}\begin{Bmatrix}\epsilon_1\\ \epsilon_2\\ \cdot\\ \cdot\\ \cdot\end{Bmatrix} \tag{32}$$

which is in the same form as Eq. (23). Then, again requiring the periodicity condition $\underline{y}(T)=\underline{y}(0)$ and following through as before will result in the same relation Eq. (26) found previously. Because of the nonlinearities now present, the elements of $[Q]$ as found from Eqs. (30-32) may vary with the amplitude of the initial condition used, $\underline{y}_E(0)+\epsilon_i$. This is in contrast to the linear case where $[Q]$ remains always constant. Hence, an iterative application of Eq. (26) with a

‡A comparison of these methods as well as a similar general derivation was given by Izadpanah.[11] This was pointed out to the authors by Prof. D. A. Peters of Washington University, St. Louis, Mo.

new corrected $q_E(0)$ should be done. If the nonlinearities are not too great, convergence to the required $y_E(0)$ should be rapid.

An alternative solution for the nonlinear case has been presented by Friedmann and Kottapalli[13] which is based on expanding the nonlinear function $G(t,y,\dot{y})$ of Eq. (29) into a first-order Taylor series in y and $\dot{y}$. Then, the equations are rearranged into the standard form of Eq. (11), where now both $A(t)$ and $G(t)$ contain the nonlinear first-order derivatives of the original $G(t,y,\dot{y})$. The equations of motion are then solved iteratively, starting from the linear solution, until convergence is reached.

It should be remarked that the numerical procedure for forced response described in this section can also be used for the constant coefficient linear case, although it is probably easier there to obtain the solution by using harmonic response methods given by Eqs. (6-10). However, for cases where there is some nonlinearity, the present iterative approach becomes attractive.

Finally, it should be remarked that the forced response of the linear system Eq. (11) can also be obtained by modal decomposition as in the constant coefficient case. To do this, set

$$\underline{y}(t)=\underline{B}(t)\{\xi_k(t)\} \tag{33}$$

where $\{\xi_k(t)\}$ is a column matrix of new coordinates and $\underline{B}(t)$ is the periodic function square matrix, each of whose columns $\{\underline{B}(t)\}_k$ was given in the previous subsection as,

$$\{B(t)\}_k=[Q(t)]\{B(0)\}_k e^{-p_k t} \tag{34}$$

Placing Eq. (33) into Eq. (11), making use of Eq. (34), noting that $\dot{\underline{Q}}(t)=\underline{A}\underline{Q}(t)$, and multiplying through by $[B(t)]^{-1}$ results finally in the uncoupled normal equations of motion,

$$\dot{\underline{\xi}}-\lceil p_k \rfloor \underline{\xi}=[B(t)]^{-1}\underline{G}(t) \tag{35}$$

where $\lceil p_k \rfloor$ is now a diagonal matrix. Equation (35) can be solved in an uncoupled manner for each mode $\xi_k(t)$ and then recombined according to Eq. (33) to obtain the total response. Such an approach requires not only keeping the eigenvalues $\lambda_k=e^{p_k T}$ and eigenvectors $\{B(0)\}_k$ of the basic transition matrix $[Q]$, but also keeping the transition matrix $[Q(t)]$ at every time interval t. This modal decomposition approach was described and used by Schrage and Peters[14] for some helicopter blade response problems.

Multiblade Coordinates and Harmonic Balance

To introduce the subject of multiblade coordinates and harmonic balance, it is convenient to look at a specific problem, namely, the stability and response of a multibladed wind turbine rotor. Given is a rotor with N blades rotating with rotation speed Ω and attached to a flexible tower. The tower is assumed to have a flexible side motion $x(t)$ (i.e., horizontal and in the blades' plane of rotation), while each of the k blades is assumed to have a lag hinge at its root which allows a small rotation $\beta^{(k)}(t)$ in the blades' plane of rotation. Because the tower motion x is described in a fixed reference frame while the blade motions $\beta^{(k)}$ are described relative to a rotating frame, the resulting equations may have mass, damping, or stiffness coefficients which are functions of the azimuthal position of the kth blade ψ_k. A typical such set of equations is given, for example, in Refs. 15 and 16 as,

$$M\ddot{x}+C_x\dot{x}+k_x x+S\frac{d^2}{dt^2}\sum^{k}\beta^{(k)}\cos\psi_k=F_x(t)$$
$$S\ddot{x}\cos\psi_k+I\ddot{\beta}^{(k)}+C_\beta\dot{\beta}^{(k)}+k_\beta\beta^{(k)}=F_\beta^{(k)}(t) \tag{36}$$
$$(k=1,2,\ldots N)$$

where the azimuthal position ψ_k is,

$$\psi_k=\Omega t+(k-1)2\pi/N \tag{37}$$

and is measured counterclockwise from the vertical axis location.

The first equation of Eqs. (36) represents force equilibrium for the tower motion x, while the remaining N equations represent force equilibrium for the motion of each of the N blades $\beta^{(k)}$. The above equations are readily generalized to more tower motions x_i and more blade coordinates for each blade $\beta_i^{(k)}$.

Stability

To examine Eqs. (36) for stability, set $F_x=0$ and $F_\beta^{(k)}=0$ to obtain homogeneous equations.

For rotors with three or more blades $N\geq 3$, eliminate the periodic coefficients in these equations by introducing new multiblade coordinates $b_0(t)$, $b_{1s}(t)$, $b_{1c}(t)$, $b_{2s}(t)$,... such that

$$\beta^{(k)}=b_0(t)+b_{1s}(t)\sin\psi_k+b_{1c}(t)\cos\psi_k$$
$$+b_{2s}(t)\sin 2\psi_k+b_{2c}(t)\cos 2\psi_k+\ldots b_A(-1)^{k-1} \tag{38}$$

where the total number of coordinates b_i taken is equal to the number of blades N. Note, if $N=$ even number, then the additional mode $(-1)^{k-1}b_A$ is needed to complete the set. The above form is an equivalent modal representation of the N blades. Substituting these into Eqs. (36), multiplying the last N equations by $\sin\psi_k$, $\cos\psi_k$, 1, $\sin 2\psi_k$, $\cos 2\psi_k$,...$(-1)^{k-1}$, respectively, summing these last N equations, and noting the trigonometric identities,

$$\sin m\psi_k \sin n\psi_k=\tfrac{1}{2}\cos(m-n)\psi_k-\tfrac{1}{2}\cos(m+n)\psi_k$$

$$\sin m\psi_k \cos n\psi_k=\tfrac{1}{2}\sin(m-n)\psi_k+\tfrac{1}{2}\sin(m+n)\psi_k$$

$$\cos m\psi_k \sin n\psi_k=-\tfrac{1}{2}\sin(m-n)\psi_k+\tfrac{1}{2}\sin(m+n)\psi_k$$

$$\cos m\psi_k \cos n\psi_k=\tfrac{1}{2}\cos(m-n)\psi_k+\tfrac{1}{2}\cos(m+n)\psi_k$$

$$\sum_{k=1}^{N}\sin m\psi_k=N\sin m\psi_1 \qquad \text{for } m=N,2N\ldots$$
$$=0 \qquad \text{for } m\neq N,2N,\ldots$$

$$\sum_{k=1}^{N}\cos m\psi_k=N\cos m\psi_1 \qquad \text{for } m=N,2N\ldots$$
$$=0 \qquad \text{for } m\neq N,2N,\ldots$$

$$\sum_{k=1}^{N}(-1)^{k-1}\sin m\psi_k=N\sin m\psi_1 \qquad \text{for } m=N/2,3N/2,\ldots$$
$$=0 \qquad \text{for } m\neq N/2,3N/2,\ldots$$

$$\sum_{k=1}^{N}(-1)^{k-1}\cos m\psi_k=N\cos m\psi_1 \qquad \text{for } m=N/2,3N/2,\ldots$$
$$=0 \qquad \text{for } m\neq N/2,3N/2,\ldots \tag{39}$$

results in a new set of differential equations in the variables $x,b_{1s},b_{1c},b_0,b_{2s},b_{2c},\ldots b_A$, which for $N\geq 3$ blades all now have constant coefficients, namely,

$$M\ddot{x}+C_x\dot{x}+k_x x+\frac{N}{2}S\ddot{b}_c=0$$

$$\frac{N}{2}\left[I\ddot{b}_{1s}+C_\beta\dot{b}_{1s}+(k_\beta-I\Omega^2)b_{1s}-2\Omega I\dot{b}_{1c}-\Omega C_\beta b_{1c}\right]=0$$

$$\frac{N}{2}\left[S\ddot{x}+2\Omega I\dot{b}_{1s}+\Omega c_\beta b_{1s}+I\ddot{b}_{1c}+c_\beta\dot{b}_{1c}+(k_\beta-I\Omega^2)b_{1c}\right]=0$$

$$N\left[I\ddot{b}_0+C_\beta\dot{b}_0+k_\beta b_0\right]=0$$

$$\frac{N}{2}\left[I\ddot{b}_{2s}+C_\beta\dot{b}_{2s}+(k_\beta-4I\Omega^2)b_{2s}-4\Omega I\dot{b}_{2c}-2\Omega C_\beta b_{2c}\right]=0$$

$$\vdots$$

$$N\left[I\ddot{b}_A+C_\beta\dot{b}_A+k_\beta b_A\right]=0 \tag{40}$$

These equations may then be investigated for stability using the standard constant coefficient techniques described earlier. It should be noted that because of the form of eqs. (36), the above equations uncouple into several smaller groups, and in fact only the first three are coupled and involve the tower motion variable x. For additional details and applications of multiblade coordinates, see Hohenemser and Yin[17] and Johnson.[2] Multiblade coordinates were originally introduced by Coleman and Feingold[18] in their studies of helicopter ground resonance. A slightly modified unit magnitude form of multiblade coordinates has recently been suggested by Prussing[19] which consists of multiplying b_0 and b_A in Eq. (38) by $\sqrt{1/N}$ and the remaining b_{is} and b_{ic} by $\sqrt{2/N}$.

For rotors with two blades, $N=2$, the analysis is more difficult because the rotor disk no longer has polar symmetry. For this case, one may use multiblade coordinates together with harmonic balance methods to arrive at approximate solutions. The multiblade transformation of Eq. (38) for two blades can be expressed in terms of coordinates $b_T(t)$ and $b_A(t)$ as,

$$\beta^{(1)}=b_T+b_A, \qquad \beta^{(2)}=b_T-b_A \tag{41}$$

Introducing these coordinates into Eqs. (36) and then summing and subtracting the last two equations of Eqs. (36), respectively, while noting that $\sin\psi_2=-\sin\psi_1$ and $\cos\psi_2=-\cos\psi_1$, results in a new set of differential equations in the variables x, b_T, and b_A which still have periodic coefficients. Then, expand each of the coordinates in a harmonic series,

$$x=x_0+x_{1S}\sin\Omega t+x_{1C}\cos\Omega t+x_{2S}\sin 2\Omega t+\ldots$$

$$b_T=b_{T0}+b_{T1S}\sin\Omega t+b_{T1C}\cos\Omega t+b_{T2s}\sin 2\Omega t+\ldots$$

$$b_A=b_{A0}+b_{A1S}\sin\Omega t+b_{A1C}\cos\Omega t+\ldots \tag{42}$$

where $x_0,x_{1S},b_{T0},b_{T1S},\ldots$ are all functions of time. Placing these functions into the previous equations and balancing each harmonic term in each equation will yield an infinite set of constant coefficient differential equations which can be truncated at some point for approximate solutions. These truncated equations may be examined again for stability using the standard constant coefficient techniques described earlier.

Often, depending on the form of Eqs. (36), the resulting constant coefficient differential equations will uncouple into several smaller coupled systems of equations which may be examined independently of one another. For example, for the case of Eqs. (36), one smaller coupled system would involve the variables $x_0,x_{2S},x_{2C},b_{A1C},b_{A1S},\ldots$ while another would involve $x_{1C},x_{1S},b_{A0},b_{A2C},b_{A2S},\ldots$. For such systems, an alternate extended form of the multiblade coordinate transformation of Eq. (38), could be used, namely,

$$x=x_0+x_{2S}\sin 2\Omega t+x_{2C}\cos 2\Omega t+\ldots$$

$$\beta^{(k)}=b_{1S}\sin\psi_k+b_{1C}\cos\psi_k+b_{3S}\sin 3\psi_k+\ldots \tag{43}$$

together with the harmonic balance method to solve the problem. This works here, since the form given by Eq. (43) exactly duplicates the motion of the two blades given by the general case of Eqs. (41) and (42), since $\sin\psi_2=-\sin\psi_1$, $\cos\psi_2=-\cos\psi_1$, and only $x_0,x_{2S},x_{2C},b_{A1C},b_{A1S},\ldots$ would be present. See Sheu[16] for an application of this alternate extended form of the multiblade transformation [Eq. (43)] to a simple two-bladed rotor in ground resonance [Eqs. (36)]. Solutions involving as little as three terms (x_0, b_{1s}, and b_{1c}) gave reasonable approximations to the primary instability regions. However, in more general cases [for example, if the first equation of Eqs. (36) had an additional term $M_1\ddot{x}\cos\psi_1$ or $k_1 x\cos\psi_1$ present], the resulting equations would not split into two smaller groups and the general harmonic balance method of Eqs. (41) and (42) would have to be used.

The general harmonic balance method typified by Eqs. (42) for solving equations with periodic coefficients is well known and has been used, for example, to solve Mathieu's and Hill's equations.[20] Extensive use of this method of truncating the resulting infinite determinants is also made by Bolotin in Ref. 21, although because of then-existing limitations on numerical computations, differing simple approximate solution techniques had to be used to investigate different regions of instability. By using the modern computational techniques available today, all of the eigenvalues of the truncated determinant obtained from Eqs. (42) can be solved directly and thus all of the significant instability regions can be obtained at the same time. See Takahashi[22] for application of this uniform, direct technique to coupled Mathieu equations with their many attendant simple parametric and combination resonances.

Forced Response

For rotors with three or more blades, $N\geq 3$, the multiblade coordinate transformation Eq. (38) can be used to eliminate the periodic coefficients in the basic equations of motion [Eqs. (36)], as described in the preceding subsection. The equations then reduce to the constant coefficient equations given by Eqs. (40), only now the right-hand sides are

$$\mathrm{RHS}=\begin{Bmatrix} F_x(t) \\ \sum_{k=1}^{N} F_\beta^{(k)}(t)\sin\psi_k \\ \sum_{k=1}^{N} F_\beta^{(k)}(t)\cos\psi_k \\ \sum_{k=1}^{N} F_\beta^{(k)}(t) \\ \sum_{k=1}^{N} F_\beta^{(k)}(t)\sin 2\psi_k \\ \vdots \\ \sum_{k=1}^{N} (-1)^{k-1}F_\beta^{(k)}(t) \end{Bmatrix} \tag{44}$$

instead of the previous value of zero. Under steady-state conditions, the tower and blade forces generally occur periodically in multiples of the rotation frequency Ω and can generally be expressed as

$$F_x(t) = F_{x0} + F_{x1S}\sin\psi_1 + F_{x1C}\cos\psi_1 + F_{x2S}\sin 2\psi_1 + \ldots$$
$$F_\beta^{(k)}(t) = F_{\beta 0} + F_{\beta 1S}\sin\psi_k + F_{\beta 1C}\cos\psi_k + \ldots \quad (45)$$

where $\psi_k = \Omega t + (k-1)2\pi/N$. Placing the above forces into Eqs. (44) and using the trigonometric identities and summations given by Eqs. (39), the right-hand sides of Eqs. (40) are obtained in terms of either constants or various harmonic functions of $m\Omega t$. The forced responses $x(t), b_{1s}(t), b_{1c}(t), b_0(t), b_{2s}, \ldots b_A$ can then be found using the standard techniques for constant coefficient systems discussed previously. It should be noted that because of the multiblade transformation Eq. (38), the resulting responses for the tower motion and blade motions corresponding to a constant term on the right-hand side would be x = const and $\beta^{(k)}$ given by Eq. (38), while the motions corresponding to an mth harmonic $\omega_m = m\Omega$ on the right-hand side would be of the form

$$x = x_R\cos\omega_m t - x_I\sin\omega_m t$$
$$\beta^{(k)} = b_{0R}\cos\omega_m t - b_{0I}\sin\omega_m t$$
$$+ (b_{1SR}\cos\omega_m t - b_{1SI}\sin\omega_m t)\sin\psi_k$$
$$+ (b_{1CR}\cos\omega_m t - b_{1CI}\sin\omega_m t)\cos\psi_k$$
$$+ (b_{2SR}\cos\omega_m t - b_{2SI}\sin\omega_m t)\sin 2\psi_k + \ldots \quad (46)$$

The tower thus oscillates at frequency ω_m in the fixed frame, whereas the blades may oscillate at frequencies $\omega_m, \omega_m + \Omega, \omega_m - \Omega, \omega_m + 2\Omega, \omega_m - 2\Omega, \ldots$ relative to the rotating frame, depending on which coordinates b_i are excited. For example, in the case of a three-bladed rotor $N = 3$, the $F_{\beta 1C}$ term of Eq. (45) leads to a constant term on the right-hand side which excites the x, b_{1s}, and b_{1c} coordinates and results in a constant x and a $\beta^{(k)}$ frequency of Ω, while the $F_{\beta 2C}$ term leads to a $\cos 3\psi$ term on the right-hand side which again excites x, b_{1s}, and b_{1c} and results in an x frequency of 3Ω and $\beta^{(k)}$ frequencies of 2Ω and 4Ω.

For rotors with two blades $N = 2$, one can use the harmonic balance methods of the previous subsection. The steady-state periodic tower and blade forces given by Eqs. (45) are substituted into the basic equations of motion [Eqs. (36)]. The new coordinates given by Eqs. (41) are introduced, the last two blade equations are summed and subtracted, and the tower and blade motions as given by Eqs. (42) are expanded, only now the coordinates $x_0, x_{1S}, b_{T0}, b_{T1S}, b_{A0}, \ldots$ etc., are taken to be constants rather than functions of time. Harmonically balancing the various terms in each equation results in a truncated set of algebraic equations which can be solved to obtain the coordinates $x_0, x_{1s}, b_{T0}, \ldots$ etc., corresponding to the given forcing excitations $F_{x0}, F_{x1s}, F_{\beta 0}, F_{\beta 1s}, F_{\beta 1c}, \ldots$ etc. The resulting tower and blade motions are then given directly by Eqs. (42) and (41). The resulting set of algebraic equations will often uncouple into smaller coupled sets of equations which can be examined independently of one another. This procedure is similar to that for the constant coefficient forced response case of Eq. (8), except now the periodic coefficients couple the different harmonics together. Thus, the solution will consist of many harmonics $m\Omega$ even if only one forcing harmonic $F_{\beta 1s}$ is present alone.

The use of harmonic balance methods can point out various important features of the response. However, for systems with many generalized coordinates, the introduction of additional degrees of freedom for the Fourier harmonics can make the algebra tedious. Some attempts at an operational formulation for the harmonic balance equations to minimize the matrix algebra are given by Peters and Ormiston.[23] Other systematic formulations of the harmonic balance equations are given by Kvaternik and Walton[24] and Wendell.[25] In particular, the latter reference adds five Fourier components to each of three initial degrees of freedom of a two-bladed wind turbine rotor (yaw, pitch, and teeter) to obtain a total of 15 degrees of freedom. Good comparison is obtained with more conventional numerical integration results for the stability and initial transient response as well as the forced response.

Rotating Coordinates

As an addendum to the previous multiblade coordinates and harmonic balance methods, it should be mentioned that for some problems the use of rotating coordinates is also convenient. For example, in the case of a two-bladed rotor on *isotropic* tower supports (same tower mass, damping, and stiffness in two directions, x_1 and x_2), Eqs. (36) would read,

$$M\ddot{x}_1 + C_x\dot{x}_1 + k_x x_1 + S\frac{d^2}{dt^2}\sum^k \beta^{(k)}\cos\psi_k = F_{x1}(t)$$
$$M\ddot{x}_2 + C_x\dot{x}_2 + k_x x_2 - S\frac{d^2}{dt^2}\sum^k \beta^{(k)}\sin\psi_k = F_{x2}(t)$$
$$S\ddot{x}_1\cos\psi_k - S\ddot{x}_2\sin\psi_k + I\ddot{\beta}^{(k)} + C_\beta\dot{\beta}^{(k)} + k_\beta\beta^{(k)} = F_\beta^{(k)}(t)$$
$$(k = 1,2) \quad (47)$$

The tower motions in the horizontal direction x_1 and the vertical direction x_2 can then be expressed in terms of rotating coordinates ξ_1 and ξ_2 which rotate with the rotor as

$$x_1 = \xi_1\cos\Omega t + \xi_2\sin\Omega t \qquad x_2 = -\xi_1\sin\Omega t + \xi_2\cos\Omega t \quad (48)$$

where the rotation $\psi_1 = \Omega t$ is taken from the x_2 axis toward the x_1 axis. Placing these equations into Eqs. (47), multiplying the first two equations by $\cos\psi_1$ and $\sin\psi_1$, respectively, and subtracting, then multiplying the first two equations by $\sin\psi_1$ and $\cos\psi_1$ and adding, then subtracting the third and fourth equations, and finally adding the third and fourth equations will result in a new set of differential equations in the variables ξ_1, ξ_2, b_A, b_T which now all have constant coefficients, namely,

$$M(\ddot{\xi}_1 + 2\Omega\dot{\xi}_2 - \Omega^2\xi_1) + C_x(\dot{\xi}_1 + \Omega\xi_2) + k_x\xi_1$$
$$+ 2S(\ddot{b}_A - \Omega^2 b_A) = F_{x1}\cos\Omega t - F_{x2}\sin\Omega t$$
$$M(\ddot{\xi}_2 - 2\Omega\dot{\xi}_1 - \Omega^2\xi_2) + C_x(\dot{\xi}_2 - \Omega\xi_1) + k_x\xi_2$$
$$- 4S\Omega\dot{b}_A = F_{x1}\sin\Omega t + F_{x2}\cos\Omega t$$
$$2S(\ddot{\xi}_1 + 2\Omega\dot{\xi}_2 - \Omega^2\xi_1) + 2I\ddot{b}_A + 2C_\beta\dot{b}_A + 2k_\beta b_A$$
$$= F_\beta^{(1)} - F_\beta^{(2)}$$
$$2I\ddot{b}_T + 2C_\beta\dot{b}_T + 2k_\beta b_T = F_\beta^{(1)} + F_\beta^{(2)} \quad (49)$$

In the above, $b_T = [\beta^{(1)} + \beta^{(2)}]/2$ and $b_A = [\beta^{(1)} - \beta^{(2)}]/2$ are the same coordinates introduced earlier in Eqs. (41). These differential equations may then be investigated for stability and forced response using the standard constant coefficient techniques described earlier. Such analyses of a two-bladed rotor on isotropic tower supports were also performed by Coleman and Feingold[18] in their studies of helicopter ground resonance.

Rotating coordinates are often used in rotating machinery shaft critical speed problems and are useful for dealing with problems of rotors with unsymmetrical mass, unsymmetrical damping, or unsymmetrical shaft stiffness supported on isotropic bearings. For example, see Bolotin.[26] For such problems, one can readily set up the equations of motion in the rotating frame directions and the fixed supports will introduce no periodic terms because of their isotropic nature. For vertical axis wind turbines, such rotating coordinates for the blades are useful since the tower supports are generally isotropic due to the symmetrically arranged guy wires. See Carne et al.[27] For horizontal axis wind turbines, the tower supports are generally not isotropic; hence, periodic coefficients will remain in the equations when using rotating coordinates.[10] If the support anisotropy is not too large, harmonic balance methods can be introduced to eliminate the periodic coefficients, as was done in the previous section. See also Zvolanek.[3]

Perturbation Methods

For the sake of completeness, brief reference will be made to a third category of analysis procedures, namely, perturbation methods, although these will not be discussed in detail here. These perturbation methods are based on the assumption that the periodic coefficients are small in some sense and that an expansion can be made in terms of some small parameter ϵ. A good discussion of these methods can be found, for example, in the book of Nayfeh and Mook.[5]

Perturbation methods have had use in the past as approximate solutions because of their simpler computational requirements. However, they require more knowledge of the system behavior, and each instability region of the system need be investigated by a separate analysis. A basic formulation of the perturbation method for periodic coefficient systems is given by Hsu.[28] Extensive use of these methods is made by Bolotin[21] for solving many physical structural dynamic problems involving Mathieu's and Hill's equations. Multiple time-scale approaches are discussed and applied by Nayfeh and Mook[5] as well as by Tong[29] and Johnson.[30] The multiple time-scale approach has also been generalized to systems of equations so that little algebra is necessary. For example, see Wei and Peters[31] which gives a matrix formulation that includes two expansion terms. Dreier[32] presents a perturbation method for the MOSTAS helicopter blade computer code which includes forced response as well as stability.

Conclusion

This paper has reviewed two of the more common procedures for analyzing the stability and forced response of rotating systems with periodic coefficients, namely, Floquet methods and multiblade coordinate, harmonic balance methods. Also, the use of rotating coordinates and perturbation methods were briefly discussed. The paper focused on the analytical techniques involved. Details of the computational aspects of the methods reviewed are cited in the references.

The Floquet methods are based on any convenient numerical integration scheme and involves the computation of the "transition matrix" $[Q]$ from which stability and the initial conditions for steady-state response solutions can be obtained. These methods seem attractive for large systems and can be modified to include nonlinearities in the equations.

The multiblade and harmonic balance methods involve first the introduction of multiblade coordinates in order to take out the periodic coefficients from the blades [Eq. (38) for $N \geq 3$] or to obtain a better ordered system of equations [Eqs. (41) for $N=2$]. Then harmonic balance methods of Eqs. (42) are used to deal with any remaining periodic coefficients. These methods seem attractive for smaller systems and can give considerable insight into the origin and nature of instabilities and the various harmonics present in the forced response. Systematic formulations of the harmonic balance method have been developed, particularly for the steady-state forced response.

Rotating coordinates can also be used to effectively eliminate the periodic coefficients in problems involving unsymmetrical rotors on isotropic tower supports. These can often be used in rotating shaft critical-speed problems and for vertical axis wind turbines. If the support anisotropy is not too large, harmonic balance methods may additionally be used to deal with any remaining periodic coefficients.

Perturbation methods have had use in the past because of their simpler computational requirements. However, they require more knowledge of the system behavior and care in application.

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